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G. D'Agostino[(1)]

(1) Istituto Nazionale di Ricerca Metrologica (INRIM), c/o University of Pavia, via Taramelli 12, 27100 Pavia, Italy

Email of the corresponding author: g.dagostino@inrim.it

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Abstract: The signal recorded in absolute ballistic gravimeters and used to track the trajectory of the free-falling corner-cube was recently studied for the first time in full agreement with relativistic effects. The outcome was an original phase-time equation describing the optical interference as a function of initial position, initial speed, dimensions and refractive index of the corner-cube. This offers the unique possibility to check possible systematic errors affecting data collected using the conventional position-time equation coupled to the debated "speed of light" correction. To this aim, I simulated and processed threshold-crossing times anticipated by the phase-time equation in typical experimental conditions of working gravimeters. In addition, the fictitious condition when a mirror replaces the corner-cube is investigated to probe the significant correction suggested by the developer of the phase-time equation. The results are given in terms of position and gravity acceleration errors.

1. Introduction

The fringe signal produced in absolute ballistic gravimeters and used to measure the center of mass position of the free-falling corner-cube (CC) retroreflector was recently described in a detailed analysis based on relativistic principles [1].

The plumb-line aligned measurement beam was tracked from the beamsplitter, through the glass of the flying CC and back down to the beamsplitter where it is recombined with the reference beam to form the interference fringes. The outcome was a "relativistic" equation describing the motion in terms of phase difference at the point of recombination.

In addition, threshold-crossing times of fringes collected with a direct free-fall type gravimeter were processed in a least-square fitting adopting the proposed phase-time equation to determine the gravity value. The result was compared with the value obtained by reprocessing the data with the same equation adjusted to prevent the measurement beam entering the CC, i.e. when it degenerates into a mirror. The difference was significant and suggested as a correction of the gravity value measured with the (conventional) "non-relativistic" position-time equation.

The subject was extensively discussed in several comments and replies without reaching a final agreement [2-7].

In this study, I follow a different approach by assuming that the phase-time equation, developed in full respect of the relativistic theory, is correct. Accordingly, I use the equation to compute the threshold-crossing times of fringes expected in the case of a known gravity value. The simulated threshold-crossing times are corrected for the delays due to finite light speed and processed in a least-

square fitting adopting the conventional position-time equation to determine the error of the reconstructed CC position and the error of the estimated gravity value.

The results are reported in this paper, both in the case of a direct free-fall and symmetric rise-and-fall type gravimeter.

2. Method

I consider Z a positive upwards position coordinate of the center of mass of CC whose axis is aligned with the plumb-line and the origin is located at the beamsplitter, i.e. where the measurement beam recombines with the reference beam.

In case of a vertical gravity gradient, γ, the equation of motion during the free-falling is

$$\ddot{Z} = -g + \gamma Z, \tag{1}$$

where g is the gravity acceleration at $Z = 0$. The solution, by keeping only linear terms in γ, is

$$Z(T) = Z_0 + V_0 T - \frac{1}{2} g T^2 + \gamma \left(\frac{1}{2} Z_0 T^2 + \frac{1}{6} V_0 T^3 - \frac{1}{24} g T^4 \right), \tag{2}$$

where Z_0 and V_0 are position and velocity at time $T = 0$.

The "relativistic" equation 33 reported in [1] and expressing the output signal of the interferometer in terms of phase difference versus T at $Z = 0$, i.e. at the beamsplitter, is:

$$\begin{aligned} \varphi(T) = {} & \frac{2(Dn-d)\Omega}{c} + \frac{2\Omega Z_0}{c} - \frac{2(Dn-d)\Omega V_0}{c^2} - \frac{2\Omega Z_0 V_0}{c^2} + T\left(\frac{2\Omega V_0}{c} - \frac{2\Omega V_0^2}{c^2} + \frac{2\Omega g(Dn-d)}{c^2} + \frac{2g\Omega Z_0}{c^2}\right) \\ & + T^2\left(-\frac{g\Omega}{c} + \frac{3g\Omega V_0}{c^2}\right) - \frac{g^2\Omega T^3}{c^2} + \gamma\Omega\left[T\left(-\frac{2(Dn-d)Z_0}{c^2} - \frac{2Z_0^2}{c^2}\right) + T^2\left(\frac{Z_0}{c^2} - \frac{(Dn-d)V_0}{c^2} - \frac{4V_0 Z_0}{c^2}\right),\right. \\ & \left. + T^3\left(\frac{V_0}{3c} + \frac{g(Dn-d)}{3c^2} - \frac{4V_0^2}{3c^2} + \frac{7gZ_0}{3c^2}\right) + T^4\left(-\frac{g}{12c} + \frac{5gV_0}{4c^2}\right) - \frac{g^2T^5}{4c^2}\right] \end{aligned} \tag{3}$$

where D is the CC depth, d is the distance of the center of mass of CC from its face, n is the refractive index of the CC glass, c is the light speed and $\Omega = \frac{2\pi c}{\lambda}$ is the angular frequency of the beam with λ wavelength.

The threshold-crossing time of the i-th fringe, T_i, is obtained by fixing values of g, Z_0, V_0, D, d, n and λ, and solving the equation

$$\varphi(T_i) - \varphi_0 = 2\pi\, i, \tag{4}$$

where φ_0 is the phase difference at $T = 0$.

In a working gravimeter, the center of mass of CC is made coincident with the optical center, i.e. $d = D/n$, to avoid centripetal effects due to rotations. In addition, a total number of threshold-crossing times, N, are selected every h fringes after discarding the first m fringes.

To reconstruct the trajectory, I consider $\bar{Z}$ a positive upwards coordinate of CC whose axis is aligned with the Z axis and the origin is located at Z_0, taken as the position corresponding to the release. It is worth noting that, according to (1), the gravity acceleration at Z_0 is $g - \gamma Z_0$.

The accuracy required to reconstruct the CC flying position does not allow the application of the widespread assumption that a half-wavelength distance corresponds to one fringe. Accordingly, due to the finite value of the light speed, T_i requires a correction to account for time delay variations during the trajectory and make one fringe increment equal to half-wavelength displacement. I applies the formula reported in [8]

$$T_i' = T_i - \frac{\bar{Z}(T_i')}{c}, \tag{5}$$

where T_i' is the corrected threshold-crossing time and $\bar{Z}(T_i') = Z(T_i') - Z_0 = -\frac{1}{2}\lambda i$ is the corresponding $\bar{Z}$ coordinate.

In agreement with (2), the $(\bar{Z}(T_i'), T_i')$ values are processed in a least-square fit adopting the (conventional) position-time relationship

$$\bar{Z}(T_i') = \bar{Z}_0 + \bar{V}_0 T_i' - \frac{1}{2}\bar{g}T_i'^2 + \gamma\left(\frac{1}{2}\bar{Z}_0 T_i'^2 + \frac{1}{6}\bar{V}_0 T_i'^3 - \frac{1}{24}\bar{g}T_i'^4\right) + \varepsilon_i, \tag{6}$$

to estimate $\bar{g}$ (at $\bar{Z} = 0$), $\bar{Z}_0$ and $\bar{V}_0$ (at $T' = 0$), and the position error ε_i. Eventually, I calculate the error affecting the $\bar{g}$ value with the formula

$$\Delta g = \bar{g} - (g - \gamma Z_0). \tag{7}$$

3. Simulation and data processing

I simulated and processed data mimicking effective operational conditions of gravimeters adopting the direct free-fall and symmetric rise-and-fall method.

Therefore, I fixed g = 9.8 ms^{-2}, γ = 3.0724615 × 10^{-6} s^{-2}, D = 0.0175 m, n = 1.52, λ = 6.3299121258 × 10^{-7} m and assumed Z_0 = 0.4 m, V_0 = 0 ms^{-1}, N = 1200, h = 1000, m = 60000 in the direct case whereas Z_0 = -0.15 m, V_0 = 1.39731637246675 ms^{-1}, N = 510 (255 rise and 255 falling), h = 1000, m = 60000 in the symmetric case.

In order to probe the significant correction suggested in [1], I simulated and processed additional data in the fictitious operational condition where the corner-cube is replaced by a mirror by setting D = 0 m instead of D = 0.0175 m.

The position shifts due to the approximation achieved by the numerical solutions of (4) were randomly distributed between ±0.5 × 10^{-6} nm.

The reconstructed $\bar{Z}$ position of CC and the corresponding error ε versus the corrected threshold-crossing time T_i' are displayed in Figure 1.

According to (7), the error affecting the estimated $\bar{g}$ value in a real operational condition, i.e. D = 0.0175 m, was 7.76 × 10^{-5} μGal and 7.83 × 10^{-5} μGal (1 μGal = 1 × 10^{-8} ms^{-2}) in the direct free-fall and symmetric rise-and-fall case, respectively, whereas in the fictitious operational condition, i.e. D = 0 m, was 3.77 × 10^{-5} μGal and 1.62 × 10^{-5} μGal, respectively.

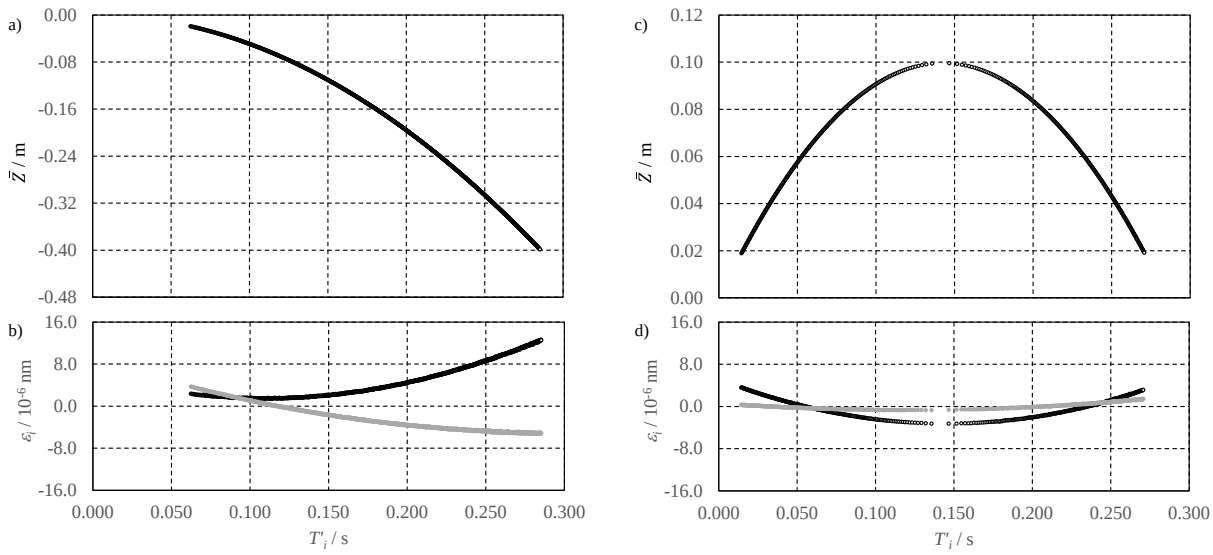

Figure 1: Reconstructed $\bar{Z}$ position of CC at T_i' and the corresponding error ε_i (black for D = 0.0175 m and gray for D = 0 m) in the case of a direct free-fall type gravimeter (a, b) and symmetric rise-and-fall type gravimeter (c, d).

3. Summary and conclusions

The phase-time equation recently developed in full agreement with the relativistic treatment of the interference between the measurement and reference beam was used to simulate the threshold-crossing times of the output fringes produced in an absolute ballistic gravimeter. Data were generated and processed by simulating actual and fictitious experimental conditions when the retroreflector is a corner-cube and a mirror, respectively.

The systematic trend observed in the reconstructed CC position pointed out that the optical path of the measurement beam within and outside the CC has relativistic effects in the output signal that are not fully accounted for. Nevertheless, the magnitude in terms of position bias was at a negligible 10^{-6} nm level.

In addition, the estimated gravity acceleration was affected by a positive error at 10^{-5} μGal level, i.e. four order of magnitude lower that the best precision currently achieved with working gravimeters, in the case of both a corner-cube and a mirror.

This endorses the present suitability of (5) to correct the threshold-crossing times anticipated by the relativistic theory of the falling retroreflector gravimeter developed by Ashby [1] and contradicts the μGal level correction suggested by the same author in equations 47 and 48.

References

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