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Spiral Tessellation on the Sphere

Marco Abrate, and Fabrizio Pollastri

***Abstract*—In this paper we describe a tessellation of the unit sphere in the 3-dimensional space realized using a spiral joining the north and the south poles. This tiling yields to a one dimensional labeling of the tiles covering the whole sphere and to a 1-dimensional natural ordering on the set of tiles of the tessellation. The correspondence between a point on the sphere and the tile containing it is derived as an analytical function, allowing the direct computation of the tile. This tessellation exhibits some intrinsic features useful for general applications: absence of singular points and efficient tiles computation. Moreover, this tessellation can be parametrized to obtain additional features especially useful for spherical coordinate indexing: tiles with equal area and good shape uniformity of tiles. An application to spherical indexing of a database is presented, it shows an assessment of our spiral tiling for practical uses.**

***Index Terms*—computational geometry, surface tessellations, spherical geometry, spherical covering.**

I. Introduction

TILING theory is a broad subject that develops through geometry, topology, group theory, number theory and other branches of mathematics. This theory has a number of applications that inspired several approaches and solutions in various fields such as astronomy, meteorology, physics and database modeling.

The most widely considered spherical tessellations belong to the homohedral tiling, that is a tiling in which all tiles are congruent. In particular, triangular homohedral tessellations have been considered by Sommerville [1] and Davies [2]. Recently d'Azevedo [3] dealt with triangulations of the sphere with certain additional conditions.

A widely applied method for the tiling of the sphere, as well as other surfaces of the 3-dimensional space, is the Voronoi tessellation, inspired by the Thompson problem, that focus on the construction of set of points uniformly distributed on the sphere (see [4], [5]) and that is used to model global atmosphere dynamics [6].

Another class of tessellations of the sphere has tiles formed by regular or semiregular (spherical) polygons, obtained by projecting a polyhedron onto a circumscribed sphere from its center.

Important applications are the mosaic grids introduced in the field of meteorology, in order to overcome the difficulties associated with numerical singularities at the poles, arising when dealing with longitude-latitude based grids (see [7], [8]); the cubed-sphere grid is an example of a mosaic grid comprising of six structured grids that are assembled to form a nearly regular tiling of the sphere.

Sphere tessellations are widely employed for astronomical applications, spacecraft attitude determination and navigation, space surveillance, etc. Spatial application are generally performed via the so-called star tracking algorithms of star sensor systems (see [9]). The principle of star sensor operation is to image stars and match the observed constellation to a star catalogue. A number of algorithms have been developed, even recently, to improve the performance of the process of star pattern recognition [10]–[13], in terms of both accuracy and speed. For star identification by star trackers, the position of the stars and the brightness are necessary. Moreover, mostly often used astrometric reference catalogues include data from about hundred of thousand up to hundred of millions stars; performance and quality of star trackers depend on execution time and success rate of matching algorithm [14], [15]. Due to this, it comes out the need to speed up star searching in huge stellar catalogues [16], [17]. An example is the cone search, where a generally small cone of view is given and the list of all stars in the catalogue seen through the cone is the expected result. Such an application requires to have indexed tables of star coordinates. A variety of sky tessellation have been proposed for the scope with various mapping functions, such as HTM and HEALPix schema, igloo pixelizations and others (see [18]–[21] for details).

In terms of database indexing, mapping a sphere with a tiling scheme leads to an efficient indexing of data [22] and, furthermore, means transforming a 2-dimensional into a 1-dimensional space. Consequently a standard B-tree index can be created on the column with the pixel IDs. On a large astronomical catalogue, this could lead to a gain of orders of magnitude in search efficiency. Being able to quickly retrieve the list of objects in a given region of the sky is crucial in several projects.

Most of tessellations in the considered applications are edge-to-edge tiling, that are tessellations where corners and sides of the polygonal tiles form all the vertices and edges of the tiling and vice versa.

If the restriction to edge-to-edge tilings is removed, a number of new tilings may be found (see Dawson and Doyle [23]–[25] for triangular case).

Following this way, this article presents a spiral tessellation which aims to create a 1-dimensional natural sort order on the tiles, without the constraint of edge-to-edge tessellation. This tessellation exhibits, as an intrinsic feature, efficiency in tiles computation, which is very useful for general applications. Moreover, it can be parametrized to obtain additional features especially useful for spherical coordinate indexing:

1) good shape uniformity of tiles,
2) tiles with equal area,
3) an efficient database building for coordinate indexing,
4) efficient search.

II. The spiral tessellation

The spiral tessellation on the unit sphere is obtained by a spiral curve with constant slope starting from the zenith

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point of the sphere (north pole), winding on the sphere and ending to the opposite point (south pole). This spiral bounds a stripe that covers the whole sphere surface. The tessellation is created by stripe segmentation with two constraints applied to all tiles, except for the first and the last tiles: equal area tiles and good shaped tiles that is tile height approximately equal to mean tile width. The basic idea is to renounce to the edge-to-edge constraint in order to get a 1-dimensional natural ordering descending from the labeling of the tiles. In this section the mathematical description of the spiral tessellation is given.

The spherical coordinate system on the unit sphere was chosen having θ as longitude angle and ϕ as latitude angle. The latitude angle is preferred against the zenith angle (colatitude) for the presence of its direct equivalent in applications in the field of coordinate indexing.

Given a number n of turns that wrap around the surface S^2 of unit sphere, consider the spiral γ given by the following parametric representation:

$$\gamma(t) = \begin{cases} x = \cos(t)\cos(n\pi + 2nt) \\ y = \cos(t)\sin(n\pi + 2nt) \\ z = -\sin(t) \end{cases} \qquad t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]. \tag{1}$$

Such a spiral has the north and the south poles as the starting and the ending points respectively, and the distance between two points at the same longitude is a multiple of π/n.

A point P on the spiral having spherical coordinates (θ_P, ϕ_P) is identified by the corresponding parameter t_P through the following transformations (see fig. 1):

$$\theta_P \equiv n\pi + 2nt_P \pmod{2\pi}, \ \phi_P = -t_P. \tag{2}$$

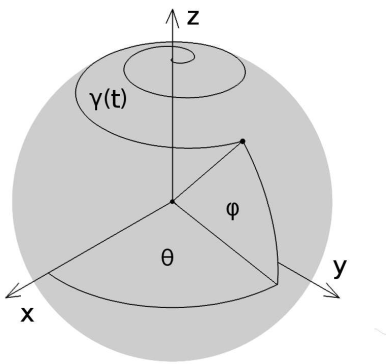

Fig. 1. The spiral and its spherical coordinates.

Definition 1. *The spiral tessellation of the sphere S^2 is the set of tiles consisting of portions of S^2 delimited by two consecutive turns of the spiral given by equation (1) (upper and lower edges) and two fixed sections of meridians (left and right edges).*

The ordering on the parameter interval $[-\pi/2, \pi/2]$ induces a natural order on the set of tiles covering S^2: suppose $M \in \mathbb{N}$ is the number of tiles in the spiral tessellation and let $\gamma(t)$ be the closest vertex to the north pole of the tile T. This point is unique by equation (2), so that the ordered set $\mathcal{T} = \{t_1, \cdots, t_{M-1}\}$ can be constructed to identify the tessellation. Therefore we can denote by T_i the tile having $\gamma(t_i)$ as higher vertex.

As a convention assume that for $0 < i < M-1$, tile T_i contains the portion of the spiral $\gamma([t_i, t_{i+1}[)$, the points having the same longitude of $\gamma(t_i)$, the points lying between $\gamma(t_i)$ and $\gamma(t_i + \pi/n)$ (except $\gamma(t_i + \pi/n)$). In common words, the upper and left edges belong to the tile, while the lower and right edges do not. Observe that every tile but T_0 and T_{M-1} have four edges, while T_0 and T_{M-1} have two edges (see figure 2). This allows for a tessellation without gaps and overlaps. It is to be noted that the first segmentation of the spiral stripe is at the left edge of T_1, that is at $t_1 = -\pi/2$, while the last one is at the right edge of T_{M-1}, and this happens at $t = (n-2)\pi/2n$.

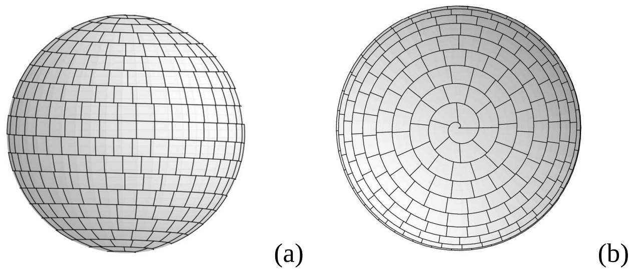

Fig. 2. The spiral tessellation with $n = 20$ and $m = 510$ (side view (a) and top view (b)).

In Definition 1 nothing is said about the distribution of values t_i. In applications, tiles are often asked to be uniformly distributed on the sphere. Nevertheless, the length of spirals on spheres is described by elliptic integrals (see [26], [27]), so that a tiling based on equally spaced vertices $\gamma(t_i)$ could be difficult to work with.

We start analyzing the properties of the spiral tiling by the area calculation of a generic tile of the grid. Denoting by $A(T_i)$ the area of the $i-th$ tile, we have

$$\begin{aligned} A(T_i) &= \int_{T_i} \mathrm{d}T_i = \int_{\theta_i}^{\theta_{i+1}} \int_{\phi(\theta+2\pi)}^{\phi(\theta)} \cos\phi \, \mathrm{d}\phi \, \mathrm{d}\theta = \\ &= \int_{n\pi+2nt_i}^{n\pi+2nt_{i+1}} \int_{\frac{\pi}{2}-\frac{\theta+2\pi}{2n}}^{\frac{\pi}{2}-\frac{\theta}{2n}} \cos\phi \, \mathrm{d}\phi \, \mathrm{d}\theta = \\ &= 2\sin\left(\frac{\pi}{2n}\right) \int_{n\pi+2nt_i}^{n\pi+2nt_{i+1}} \sin\left(\frac{\theta+\pi}{2n}\right) \mathrm{d}\theta = \\ &= 4n\sin\left(\frac{\pi}{2n}\right)\cos\left(t_i + \frac{(n+1)\pi}{2n}\right) - \\ &\quad\ 4n\sin\left(\frac{\pi}{2n}\right)\cos\left(t_{i+1} + \frac{(n+1)\pi}{2n}\right). \end{aligned}$$

Let Σ be the portion of S^2 covered by the four-edged tiles, and let $m = M - 2$ be the number of tiles covering Σ.

By previous calculation, the area of the portion of Σ up to the $(i-1)-th$ tile is

$$\begin{aligned} \sum_{k=1}^{i-1} A(T_k) &= \int_{n\pi+2nt_1}^{n\pi+2nt_i} \int_{\frac{\pi}{2}-\frac{\theta+2\pi}{2n}}^{\frac{\pi}{2}-\frac{\theta}{2n}} \cos\phi \, \mathrm{d}\phi \, \mathrm{d}\theta = \\ &= 4n\sin\left(\frac{\pi}{2n}\right)\cos\left(\frac{(n+1)\pi}{2n}\right) - \\ &\quad\ 4n\sin\left(\frac{\pi}{2n}\right)\cos\left(t_i + \frac{(n+1)\pi}{2n}\right), \end{aligned}$$

hence the area of Σ is

$$A(\Sigma)=\int_0^{2(n-1)\pi}\int_{\frac{\pi}{2}-\frac{\theta+2\pi}{2n}}^{\frac{\pi}{2}-\frac{\theta}{2n}}\cos\phi\,\mathrm{d}\phi\,\mathrm{d}\theta=4n\sin\left(\frac{\pi}{n}\right).$$

For our purpose, let us consider a tessellation where all the tiles covering Σ have the same area $A=A(T_i)$: it follows that

$$A=\frac{4n}{m}\sin\left(\frac{\pi}{n}\right), \tag{3}$$

so that

$$\sum_{k=1}^{i-1}A(T_k)=\frac{4n(i-1)}{m}\sin\left(\frac{\pi}{n}\right).$$

By direct calculation, we can derive the equation for the parameters which identify the vertices for the tiling:

$$t_i=\arccos\left(\cos\left(\frac{\pi}{2n}\right)\left(1-\frac{2(i-1)}{m}\right)\right)-\frac{(n+1)\pi}{2n}. \tag{4}$$

If the number n of turns and the number m of tiles are chosen to tessellate the spheric portion Σ, equation (4) determines the set $\mathcal{T}=\{t_1,\cdots,t_{m+1}\}$ identifying the tessellation.
The sphere surface S^2 is then tiled by $m+2$ tiles: m of them cover Σ and have all the same area $\frac{4n}{m}\sin\left(\frac{\pi}{n}\right)$, and the remaining tiles are the two-edged tiles, namely T_0 and T_{m+1}. T_0 is above the first turn of γ: it does not contain any point of the border because such points belong to the tiles that are right below (in particular the north pole belongs to T_1). T_{m+1} is below the last turn of γ: this set contains all the point on its boundary (as well as the south pole).
Since $A(S^2)=4\pi$, both T_0 and T_{m+1} have area

$$A(T_0)=A(T_{m+1})=2\pi-2n\sin\left(\frac{\pi}{n}\right). \tag{5}$$

A. A nearly uniform distributed spiral tiling

In order to build a nearly uniform distributed tiling, the equal surface constraint has been introduced on four-edges cells.
Going further on the nearly uniformity goal, good shaped tiles are desired, so the requirement for A to be close to the square of the slope of the spherical spiral is added. We want to evaluate n and m fixing the desired value for A, namely $\hat{A}$. The number of spiral windings on the sphere can be computed by dividing the parametrization set of t by the square root of $\hat{A}$, that is

$$n=\frac{\pi}{\hat{A}^{1/2}}. \tag{6}$$

It should be noted that, to obtain a symmetric tiling, it can be useful to consider an even number of spiral windings. However, no constraints on the number n is needed for the realization of the tiling, so that n can be any real number greater than 1.

Substituting (6) in equation (3) we have that for $i=1,...,m$

$$A(T_i)=A=\frac{4n}{m}\sin\left(\frac{\pi}{n}\right)=\frac{4\pi}{m\hat{A}^{1/2}}\sin\left(\hat{A}^{1/2}\right)$$

and an expression for m is given by

$$m=\frac{4\pi}{A\,\hat{A}^{1/2}}\sin\left(\hat{A}^{1/2}\right).$$

Setting

$$m=\left\lceil\frac{4\pi}{\hat{A}^{3/2}}\sin\left(\hat{A}^{1/2}\right)\right\rceil. \tag{7}$$

it follows

$$\frac{4\pi}{\hat{A}^{3/2}}\sin\left(\hat{A}^{1/2}\right)\le m<\frac{4\pi}{\hat{A}^{3/2}}\sin\left(\hat{A}^{1/2}\right)+1.$$

and the actual value of A is such that

$$\hat{A}\cdot\frac{4\pi\sin(\hat{A}^{1/2})}{4\pi\sin(\hat{A}^{1/2})+\hat{A}^{3/2}}<A\le\hat{A}.$$

This implies that the difference between $\hat{A}$ and A is bounded by

$$0\le\hat{A}-A<\frac{\hat{A}^{5/2}}{4\pi\sin(\hat{A}^{1/2})+\hat{A}^{3/2}}=\frac{\hat{A}^2}{4\pi}+o(\hat{A}^2). \tag{8}$$

By equation (8) we get the following approximation for A:

$$\hat{A}-\frac{\hat{A}^2}{4\pi}+o(\hat{A}^2)<A\le\hat{A}. \tag{9}$$

The area of the four-edged tiles can be compared with $A(T_0)$ and $A(T_{m+1})$ assuming that n and m are calculated by equations (6) and (7).
By equation (5) we have

$$\begin{aligned}A(T_0)&=2\pi-2\frac{\pi}{\hat{A}^{1/2}}\sin\left(\hat{A}^{1/2}\right)=\\&=\frac{2\pi}{\hat{A}^{1/2}}\left(\hat{A}^{1/2}-\sin\left(\hat{A}^{1/2}\right)\right),\end{aligned}$$

so that the ratio between $A(T_0)$ and A, setting $A=\hat{A}+o(\hat{A}^{3/2})$ according to equation (9), is given by

$$\begin{aligned}\frac{A(T_0)}{A}&=\frac{2\pi}{A\hat{A}^{1/2}}\left(\hat{A}^{1/2}-\sin\left(\hat{A}^{1/2}\right)\right)=\\&=\frac{2\pi}{\hat{A}^{3/2}+o(\hat{A}^2)}\cdot\frac{\hat{A}^{3/2}+o(\hat{A}^2)}{6}=\frac{\pi}{3}+o(\hat{A}).\end{aligned}$$

From this result, it comes out that both the polar tiles have a surface that is very close to the four-edges tiles surface: it is less than 5% bigger.

III. CORRESPONDENCE POINT-TILE

For query execution acceleration purpose, we intend to induce a raster structure on the database based on the sphere tessellation introduced in section II. This process can be allowed by calculating the belonging tile of data stored. The result is a new database where data are aggregated by tile.
In the next theorem we describe the correspondence between the set of points on the sphere, represented by his canonical latitude-longitude coordinates, and the set of the tiles labels. This relation yields to a direct and fast localization of a point in the grid.

Theorem 1. *Let $P=P(\theta,\phi)$ be a point on the unit sphere. The index i of the tile T_i of P is given by*

$$i=\left\lfloor\frac{m\left(\cos\left(\frac{\pi}{2n}\right)-\cos\left(\tilde{t}+\frac{(n+1)\pi}{2n}\right)\right)}{2\cos\left(\frac{\pi}{2n}\right)}\right\rfloor+1$$

where $\tilde{t}=\dfrac{\theta}{2n}+\dfrac{\pi}{n}\left\lfloor\dfrac{n\pi-\theta-2n\phi}{2\pi}\right\rfloor-\dfrac{\pi}{2}.$

Proof: In order to determinate the tile the point belongs to, we find the point on the spiral having the same longitude θ sited in the upper edge of the tile of P. Such a point is given by the parametrization of the spiral for $t = \tilde{t}$, that is $\gamma(\tilde{t}) = \tilde{P}(\tilde{\theta}, \tilde{\phi})$.
The point $\gamma(\tilde{t})$ is the intersection between the meridian trough P and the portion of spiral for $t \in [t_i, t_{i+1}[$. If $k \in \mathbb{N}$ is the number of windings of the spiral above P,

$$\tilde{\phi} = \frac{\pi}{2} - \frac{\theta}{2n} - k\frac{\pi}{n}.$$

A direct computation yields to the next equation:

$$k\frac{\pi}{n} + \tilde{\phi} - \phi = \frac{\pi}{2} - \frac{\theta}{2n} - \phi.$$

Since $0 \leq \frac{n}{\pi}\left(\tilde{\phi} - \phi\right) < 1$ it follows that

$$k = \left\lfloor \frac{n\pi - \theta - 2n\phi}{2\pi} \right\rfloor. \tag{10}$$

If $0 \leq k \leq n$ we finally have

$$\tilde{\phi} = \frac{\pi}{2} - \frac{\theta}{2n} - \frac{\pi}{n}\left\lfloor \frac{n\pi - \theta - 2n\phi}{2\pi} \right\rfloor$$

and, by equation (2),

$$\tilde{t} = \frac{\theta}{2n} + \frac{\pi}{n}\left\lfloor \frac{n\pi - \theta - 2n\phi}{2\pi} \right\rfloor - \frac{\pi}{2}. \tag{11}$$

If k given by equation (10) is negative we have

$$P \in T_0,$$

while for $k \geq n - 1$

$$P \in T_{m+1}.$$

The tile index can be computed analytically by the value of $\tilde{t}$ obtained in (11) by observing that if $P \in T_i$ then $t_i \leq \tilde{t} < t_{i+1}$, that is to say that, using equation (4), $\cos\left(\tilde{t} + \frac{(n+1)\pi}{2n}\right)$ lies between $\cos\left(\frac{\pi}{2n}\right)\left(1 - \frac{2i}{m}\right)$ and $\cos\left(\frac{\pi}{2n}\right)\left(1 - \frac{2(i-1)}{m}\right)$.

It follows that

$$1 - \frac{2(i-1)}{m} \geq \frac{\cos\left(\tilde{t} + \frac{(n+1)\pi}{2n}\right)}{\cos\left(\frac{\pi}{2n}\right)} > 1 - \frac{2i}{m}$$

$$i - 1 \leq \frac{m\left(\cos\left(\frac{\pi}{2n}\right) - \cos\left(\tilde{t} + \frac{(n+1)\pi}{2n}\right)\right)}{2\cos\left(\frac{\pi}{2n}\right)} < i$$

or

$$i = \left\lfloor \frac{m\left(\cos\left(\frac{\pi}{2n}\right) - \cos\left(\tilde{t} + \frac{(n+1)\pi}{2n}\right)\right)}{2\cos\left(\frac{\pi}{2n}\right)} \right\rfloor + 1.$$

■

IV. THE NEIGHBORHOOD OF A TILE

Ordering on the tiling can be very helpful in applications, even though the lack of the edge-to-edge property leaves the problem to manage the edges and the neighborhood of a tile. Theorem 1 can be applied to identify the tiles in a neighborhood of a given tile T_i, for $0 < i \leq m$: since T_i has four vertices $\gamma(t_i)$, $\gamma(t_{i+1})$, $\gamma\left(t_{i+1} + \frac{\pi}{n}\right)$ and $\gamma\left(t_i + \frac{\pi}{n}\right)$, one has that for $t_i \geq -\frac{\pi}{2} + \frac{\pi}{n}$ the points $\gamma\left(t_i - \frac{\pi}{n}\right)$ and $\gamma\left(t_{i+1} - \frac{\pi}{n}\right)$ are in the tiles above to T_i, namely T_{d_1} and T_{d_2}, which can be found operating as described above. By the same way it can be possible to find T_{d_3} and T_{d_4} where $\gamma\left(t_{i+1} + \frac{\pi}{n}\right)$ and $\gamma\left(t_i + \frac{\pi}{n}\right)$ are in the case $t_i < \frac{\pi}{2} - \frac{2\pi}{n}$, which are immediately below T_i. Hence the neighborhood of T_i is described by the set of tiles

$$T_j, \text{ where } j = i+1,\ j = i-1,\ d_1 \leq j \leq d_2,\ d_3 \leq j \leq d_4.$$

Some attention have to be made if $t_i < -\frac{\pi}{2} + \frac{\pi}{n}$ or $t_{i+1} \geq \frac{\pi}{2} - \frac{2\pi}{n}$: in the first case the neighborhood of T_i will contain T_0, in the second one T_{m+1}. We can summarize all the cases in the following:

1) if $t_{i+1} < -\frac{\pi}{2} + \frac{\pi}{n}$ the neighborhood tiles are given by:
$$N(T_i) = \{T_0, T_j\},$$
with $j = i+1$, $j = i-1$, $d_3 \leq j \leq d_4$;
2) if $t_i < -\frac{\pi}{2} + \frac{\pi}{n}$ and $t_{i+1} \geq -\frac{\pi}{2} + \frac{\pi}{n}$ the neighborhood tiles are:
$$N(T_i) = \{T_0, T_j\},$$
with $j = i+1$, $j = i-1$, $0 \leq j \leq d_2$, $d_3 \leq j \leq d_4$;
3) if $t_i > \frac{\pi}{2} - \frac{2\pi}{n}$ we have:
$$N(T_i) = \{T_j, T_{m+1}\},$$
with $j = i+1$, $j = i-1$, $d_1 \leq j \leq d_2$;
4) if $t_i < \frac{\pi}{2} - \frac{\pi}{n}$ and $t_{i+1} \geq \frac{\pi}{2} - \frac{2\pi}{n}$ then:
$$N(T_i) = \{T_j, T_{m+1}\},$$
with $j = i+1$, $j = i-1$, $d_1 \leq j \leq d_2$, $d_3 \leq j < m$.

V. SPHERICAL DISC COVERING

In many applications to physic or to astronomical problems spherical discs are used. In this section we focus our attention on describing the requirement of a point on the sphere to belong to a given spherical disc that can be used to determine the set of tiles needed to cover such a disc.

Let $C = C(\theta_C, \phi_C)$ be a point on the sphere and let δ be an angle. C and δ can be used to describe a spherical disc centered in C with radius δ as the set of points on the surface inside the cone having vertex at the sphere center, opening angle 2δ and whose height passes trough C.

We describe a point $P = P(\theta, \phi)$ on the circumference obtained as the intersection between the sphere and the cone using some well-known facts about spherical geometry. Let us consider the spherical triangle having C, P and the north pole, namely N, as vertices. The sides from C to P and from

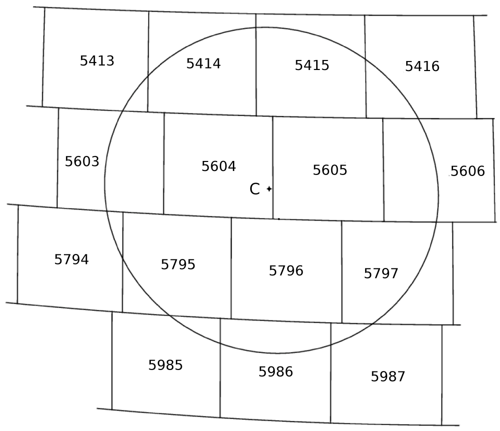

Fig. 3. An example of a cone disc centered at C and all tiles involved, each one with its own index. Tiles number 12158, spiral turns 100, cone semiangle 0.05 radians.

C to N are δ and $\frac{\pi}{2} - \phi_C$ respectively. If ψ is the angle in C, we can use the spherical law of cosines to compute its opposite side:

$$\cos\left(\frac{\pi}{2} - \phi\right) = \cos(\delta)\cos\left(\frac{\pi}{2} - \phi_C\right) + \sin(\delta)\sin\left(\frac{\pi}{2} - \phi_C\right)\cos(\psi),$$

that is

$$\sin(\phi) = \cos(\delta)\sin(\phi_C) + \sin(\delta)\cos(\phi_C)\cos(\psi). \quad (12)$$

Moreover, using the spherical law of sines, we have

$$\sin(\theta - \theta_C) = \frac{\sin(\delta)\sin(\psi)}{\cos(\phi)},$$

so that

$$\theta = \theta_C + \arcsin\left(\frac{\sin(\delta)\sin(\psi)}{\cos(\phi)}\right). \quad (13)$$

By equations (12) and (13) we get the following formulas for the computation of the points on the circumference centered in C with spherical radius δ:

$$\begin{cases} \phi = \arcsin\left(\cos(\delta)\sin(\phi_C) + \sin(\delta)\cos(\phi_C)\cos(\psi)\right) \\ \theta = \theta_C + \arcsin\left(\frac{\sin(\delta)\sin(\psi)}{\cos(\phi)}\right) \pmod{2\pi}, \end{cases}$$

where $\psi \in [0, 2\pi]$.

The same argument can be used to test whether a point of the sphere belongs to the spherical disc centered in C. By the law of cosines again we have that the arc d joining $Q = (\theta, \phi)$ to $C = (\theta_C, \phi_C)$ satisfies the following equation:

$$\cos(d) = \sin(\phi_C)\sin(\phi) + \cos(\phi_C)\cos(\phi)\cos(\theta - \theta_C).$$

Hence we derive the condition for a point to be in the disc:

$$\sin(\phi_C)\sin(\phi) + \cos(\phi_C)\cos(\phi)\cos(\theta - \theta_C) \geq \cos(\delta). \quad (14)$$

Equation (14), together with Theorem 1, provides to the set of indexes of the tiles that have to be used to cover the disc.

VI. Performance evaluation

In this section we describe an implementation of a search algorithm based on the spiral tessellation described above, in order to perform tests of performance. We report the experimental results on the performance of the search algorithm with a comparison versus a straightforward method based on a direct database search. The experiment consists of a set of cone searches performed on a star catalogue, in order to find the list of stars closer to a given pointing direction within a given radius. The Tycho-2 [28] star catalog was chosen for its relatively small size, about 2.5 million stars, to keep the tests easily performed on a normal personal computer.

Since the main parameter governing the spherical tessellation is the number of tiles, all tests were repeated for three different tile sizes having m, the total number of tiles, equal to $84\,662, 101\,595$ and $126\,994$. Such choices for m have been made in order to have about $30, 25$ or 20 objects per tile.

As a preliminary step, the star catalog is loaded into a sqlite database engine. The former step in the implemented algorithm is to modify the database containing all the star data by adding a new column, where the tile number of any star is stored. This manipulation of the database is crucial because it represents the pre-aggregation process for the database and leads to a 1-dimensional search space. When a cone search is submitted to the database, the set of tiles covering the disc is computed, according with section V, distinguishing between tiles completely contained in the disc and those covering the circumference of the disc. The stars sited in the inner tiles are selected at once by submitting a query to the database using the tile number as an index. For the border tiles, stars are selected when their coordinates satisfy the condition of equation (14).

Experiments were conducted on a personal computer with a Debian GNU/Linux environment, exhibiting a computational power of about 5300 bogomips. All required programs are coded in Python. We performed a set of 7 cone searches: cones are all centered at the same point and have radius $0.05, 0.1, 0.2, 0.5, 1.0, 1.5$ and 2.0 decimal degrees respectively. The queries are submitted in both database index methods, spherical tessellation and direct, and execution times are compared.

Figure 4 shows query execution times, expressed in CPU seconds, for cone search performed with a straightforward method and with our algorithm, implemented using three different spiral tessellations.

Our experiments show that the spiral indexing induced by our tessellation on the sphere leads to a search algorithm that is, in term of search efficiency, about 30 times faster than a straightforward search method. Moreover, the comparison is held out between a standard database query that is highly optimized and programs coded in Python that is an interpreted language 4-5 time slower than traditional lower level programming languages like C. This leads one to think that a further improvement of 4-5 times can be achieved with a further work of recoding and optimization. This result makes our search algorithm competitive with database management obtained performing other tessellations. The search speed improvement can be ascribed to both the addition of the tile index column to the star table and both accessing the data with this new combined index, reducing a bidimensional

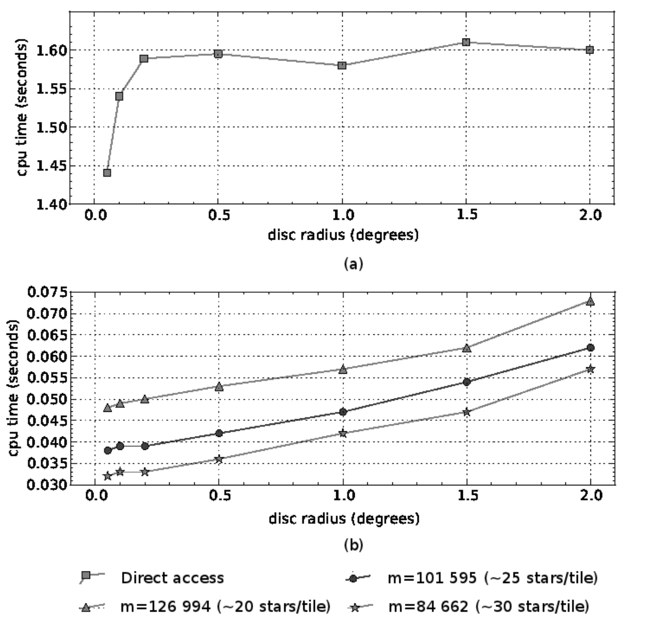

Fig. 4. Cone search execution times for a directly accessed database (a) and for a spiral tiling managed database (b) as a function of the disc radius.

search to a search made along a single dimension. By figure 4, it turns out that times grow in a quasi-linear way with respect to the disc radius and increase as m increases.

VII. Conclusion

In this paper the tessellation problem on the sphere has been dealt in a new way introducing the notion of spiral tessellation. The usual edge-to-edge condition on the tiling has been dropped out to provide a number of properties of the tiles such as the good shape uniformity and the equal surface area.

Inspired by applications to spherical coordinate indexing, we pointed out formulas to find the tile containing a given point and the set of tiles covering a given spherical disc. Such formulas have been implemented to realize a one dimensional alternative to hierarchical techniques or multiple depth search algorithm on the sphere, and an application to cone search in astronomical star catalogues has been described. The proposed search algorithm exhibits a good speed acceleration versus a highly optimized conventional database search, typically 30 times. It is conservative to suppose to achieve an extra improvement of 4-5 times on the proposed algorithm with a further work of code optimization.

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